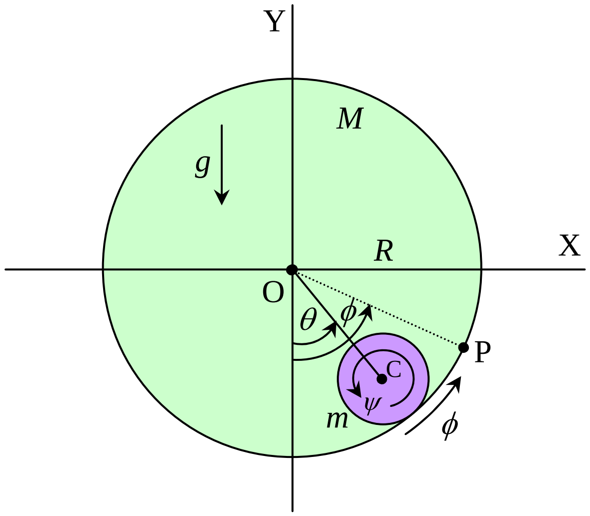
Rolling Cylinders



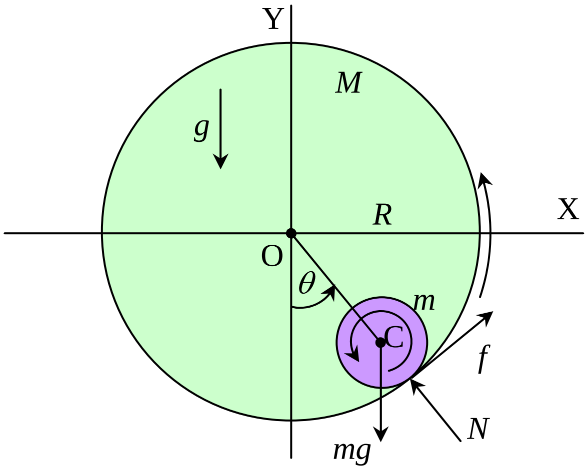
1.1) The point P which is fixed on the surface of M was at the position directly below O at time $t=0$. Hence m must have rolled through an angle $\frac{\phi R-\theta R}{r}$ radians relative to surface of M in time t during which the line OC has also turned anti-clockwise through an angle θ. Therefore the total angular displacement of m about its centre of mass relative to any fixed reference line in time t is

$$\psi = \frac{\phi R-\theta R}{r}+\theta = \frac{R}{r}\phi-\left(\frac{R-r}{r}\right)\theta \qquad \text{……….(i)} \qquad \text{(0.8 point)}$$

1.2) By differentiating the equation (i) twice with respect to time, we get

$$\frac{d^2}{dt^2}\psi = \frac{R}{r}\frac{d^2}{dt^2}\phi-\left(\frac{R-r}{r}\right)\frac{d^2}{dt^2}\theta \qquad \text{……….(ii)} \qquad \text{(0.2 point)}$$

1.3)



The equations of motion of centre of mass of m are:

$$m\left(R-r\right)\frac{d^2}{dt^2}\theta \;=\; f-mg\sin\theta \qquad \text{.............(iii)} \qquad \text{(0.7 point)}$$

$$m\left(\frac{d}{dt}\theta\right)^2\left(R-r\right) \;=\; N-mg\cos\theta \qquad \text{.............(iv)}$$

The equation for the rotation of m about its centre of mass is:

$$I_{\text{CM}}\frac{d^2}{dt^2}\psi \;=\; I_{\text{CM}}\left[\frac{R}{r}\frac{d^2}{dt^2}\phi-\left(\frac{R-r}{r}\right)\frac{d^2}{dt^2}\theta\right] \;=\; fr \qquad \text{..........(v)} \qquad \text{(0.7 point)}$$

where $I_{\text{CM}}=\frac{1}{2}mr^2$.

Equations (iii) and (v) yield:

$$\left(m+\frac{I_{\text{CM}}}{r^2}\right)\left(R-r\right)\frac{d^2}{dt^2}\theta \;=\; -mg\sin\theta+\frac{I_{\text{CM}}R}{r^2}\frac{d^2}{dt^2}\phi \qquad \text{..........(vi)} \qquad \text{(0.4 point)}$$

1.4) Here, $\frac{d^2}{dt^2}\phi=0$, $\sin\theta\approx\theta$ and also $I_{\text{CM}}=\frac{1}{2}mr^2$, the equation (vi) is reduced to:

$$\frac{d^2}{dt^2}\theta \;=\; -\frac{2g}{3(R-r)}\theta \qquad \text{..........(vii)} \qquad \text{(0.8 point)}$$

This gives a period

$$T \;=\; 2\pi\sqrt{\frac{3(R-r)}{2g}} \qquad \text{..........(viii)} \qquad \text{(0.5 point)}$$

1.5) The equilibrium position of m in question 1.4) is $\theta=0$. (0.2 point)

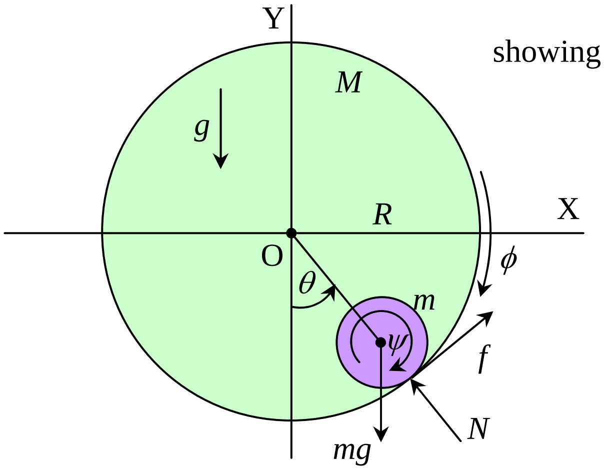
1.6) But the equilibrium position for the case where M is rotating with a constant angular acceleration α is by considering the equation (vi), namely,

$$\frac{3}{2}\left(R-r\right)\frac{d^2}{dt^2}\theta \;=\; -g\sin\theta+\frac{R}{2}\alpha \qquad \text{..........(ix)} \qquad \text{(0.4 point)}$$

Let θ_{eq} be the equilibrium position; this implies that m remains stationary at this position if it does not oscillate. Hence $\frac{d^2}{dt^2}\theta_{\text{eq}} = 0$, and (0.1 point)

$$\theta_{\text{eq}} \;=\; \arcsin\left(\frac{R\alpha}{2g}\right) \qquad \text{…………..(x)} \qquad \text{(0.2 point)}$$

1.7)



showing forces acting on *m*

From the equation (i) we get, after changing the directions of ψ and ϕ,

$$\frac{d}{dt}\psi \;=\; \frac{R}{r}\frac{d}{dt}\phi + \left(\frac{R-r}{r}\right)\frac{d}{dt}\theta \qquad \text{…………..(xi)} \qquad \text{(0.5 point)}$$

The equations of motion of m and M are:

$$\frac{1}{2}mr^2\frac{d^2}{dt^2}\psi \;=\; -fr \qquad \text{…………..(xii)} \qquad \text{(0.3 point)}$$

$$MR^2\frac{d^2}{dt^2}\phi \;=\; +fR \qquad \text{…………..(xiii)} \qquad \text{(0.3 point)}$$

Method 1: (Angular Momentum)

The effect of gravity on the system as a whole is to change its angular momentum:

$$\frac{d}{dt}\left[MR^2\frac{d}{dt}\phi + \frac{1}{2}mr^2\frac{d}{dt}\psi - m(R-r)^2\frac{d}{dt}\theta\right] \;=\; +mg(R-r)\sin\theta \qquad \text{……..(xiv.1)} \qquad \text{(0.4 point)}$$

Hence

$$\frac{d^2}{dt^2}\phi \;=\; -\frac{m(R-r)}{(2M+m)R}\frac{d^2}{dt^2}\theta \qquad \text{…………..(xv.1)}$$

and

$$\left(MR+\frac{1}{2}mr\right)R\frac{d^2}{dt^2}\phi - m(R-r)\left(R-\frac{3}{2}r\right) \;=\; mg(R-r)\sin\theta \qquad \text{……..(xvi.1)} \qquad \text{(0.4 point)}$$

Combining the last two equations:

$$\frac{d^2}{dt^2}\theta = -\frac{g}{(R-r)}\frac{(2M+m)}{(3M+m)}\sin\theta \qquad \text{..........(xvii.1)}$$

For a small-amplitude oscillation we put $\sin\theta \approx \theta$ and equation (xvii) is reduced to:

$$\frac{d^2}{dt^2}\theta = -\frac{g}{(R-r)}\frac{(2M+m)}{(3M+m)}\theta \qquad \text{..........(xviii.1)} \quad \text{(0.5 point)}$$

The period of this oscillation is, therefore,

$$T = 2\pi\sqrt{\left(\frac{R-r}{g}\right)\left(\frac{3M+m}{2M+m}\right)} \qquad \text{(0.1 point)}$$

Method 2: (Newton's law)

From Newton's law: $mg\sin\theta - f = ma$

$$mg\sin\theta - f = -m(R-r)\frac{d^2\theta}{dt^2} \qquad \text{........(xiv.2)} \quad \text{(0.4 point)}$$

From equation (xiii): $f = MR\dfrac{d^2\phi}{dt^2}$

Substitute this into equation (xiv.2) we have

$$mg\sin\theta = MR\frac{d^2\phi}{dt^2} - m(R-r)\frac{d^2\theta}{dt^2} \qquad \text{........(xv.2)}$$

From equations (xi) (xii) and (xiii), we then have

$$\frac{d^2\phi}{dt^2} = -\frac{m}{2M+m}\left(\frac{R-r}{R}\right)\frac{d^2\theta}{dt^2} \qquad \text{........(xvi.2)} \quad \text{(0.4 point)}$$

Then (xv.2) becomes $mg\sin\theta = -\dfrac{Mm}{2M+m}(R-r)\dfrac{d^2\theta}{dt^2} - m(R-r)\dfrac{d^2\theta}{dt^2}$

$$\frac{d^2\theta}{dt^2} = -\frac{g}{(R-r)}\frac{2M+m}{3M+m}\sin\theta \qquad \text{........(xvii.2)}$$

For a small-amplitude oscillation we put $\sin\theta \approx \theta$ and equation (xvii) is reduced to:

$$\frac{d^2}{dt^2}\theta = -\frac{g}{(R-r)}\frac{(2M+m)}{(3M+m)}\theta \qquad \text{........(xviii.2)} \quad \text{(0.5 point)}$$

The period of this oscillation is, therefore,

$$T = 2\pi\sqrt{\left(\frac{R-r}{g}\right)\left(\frac{3M+m}{2M+m}\right)} \qquad \text{(0.1 point)}$$

1.8) When M is made to rotate steadily at an angular velocity Ω the equation (vi) becomes

$$\frac{3}{2}(R-r)\frac{d^2}{dt^2}\theta = -g\sin\theta \qquad \text{…………..(xix)}$$

which implies that m remains at $\theta = 0$ if m does not oscillate.

Hence the equation (i) is reduced to

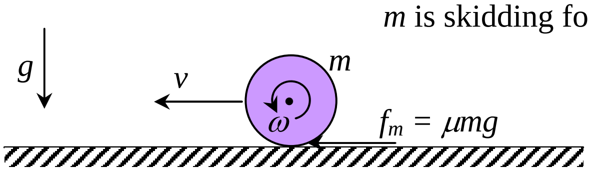
$$\psi = \frac{R}{r}\phi$$

and

$$\frac{d}{dt}\psi = \frac{R}{r}\frac{d\phi}{dt} = \frac{R}{r}\Omega \qquad \text{…………..(xx)} \qquad \text{(0.2 point)}$$

This means that m is rotating at a constant angular velocity $\frac{R}{r}\Omega$ prior to the instant when M is stopped.

After that instant m will accelerate itself by way of frictional impulse. This acceleration process lasts for only a short time due to the high value of frictional coefficient (μ). To simplify the calculation we will take to lower surface of M to be flat.



m is skidding forwards

$$m\frac{d}{dt}v = +f_m \qquad \text{…………..(xxi)}$$

$$I_{\text{CM}}\frac{d}{dt}\omega = -f_m r, \quad I_{\text{CM}} = \frac{1}{2}mr^2 \qquad \text{…………..(xxii)}$$

By solving these last two equations for $v(t)$ and $\omega(t)$ with initial conditions $v(0)=0$ and $\omega(0)=\frac{R}{r}\Omega$, and imposing the condition $v'(t)=\omega'(t)r$ for the onset of pure rolling we get

$$v' = \frac{1}{3}R\Omega, \quad \omega' = \frac{1}{3}\frac{R}{r}\Omega \qquad \text{…………..(xxiii)} \qquad \text{(0.8 point)}$$

From now on m will roll up the side of the cylindrical wall. And since frictional force does not do work in pure rolling we can use the principle of conservation of energy.

$$\frac{1}{2}mv^2+\frac{1}{2}I_{\mathrm{CM}}\omega^2+2mg\left(R-r\right) = \frac{1}{2}mv'^2+\frac{1}{2}I_{\mathrm{CM}}\omega'^2 \quad \text{…………..(xxiv)} \quad \text{(0.8 point)}$$

We have also

$$N = m\frac{v^2}{R-r}-mg \quad \text{…………..(xxv)} \quad \text{(0.2 point)}$$

$\therefore$

$$N = \left(\frac{m}{R-r}\right)\left(\frac{R\Omega}{3}\right)^2-\frac{11}{3}mg \quad \text{…………..(xxvi)} \quad \text{(0.2 point)}$$

m will reach the top if $N \geq 0$.

Hence

$$\Omega \geq \sqrt{33g\left(\frac{R-r}{R^2}\right)} \quad \text{(0.3 point)}$$

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