The Leidenfrost Phenomenon

3.1) As given in the problem

$$\frac{d}{dz}v = \left(\frac{1}{\eta}\frac{dP}{dr}\right)\cdot z \qquad \ldots\ldots\ldots\ldots\ldots \text{(i)}$$

Integrating (i) with respect to z, we get

$$v(z) = \left(\frac{1}{2\eta}\frac{dP}{dr}\right)\cdot z^2 + C \qquad \ldots\ldots\ldots\ldots\ldots \text{(ii)} \qquad \text{(0.5 point)}$$

3.2)

$$v\left(\frac{b}{2}\right) = 0 = \left(\frac{1}{2\eta}\frac{dP}{dr}\right)\cdot\left(\frac{b}{2}\right)^2 + C \qquad \ldots\ldots\ldots\ldots\ldots \text{(iii)}$$

$$\therefore \qquad C = -\frac{b^2}{8\eta}\frac{dP}{dr} \qquad \text{(0.5 point)}$$

Note that C is not a real constant; its value depends on $\frac{dP}{dr}$ which is a function of r.

3.3) Let Q be the volume rate of flow of the vapour through the cylindrical surface of $2\pi rb$.

$\delta Q = v(z)\cdot 2\pi r\delta z$ where from (ii) and (iii): (0.3 point)

$$v(z) = \left(\frac{1}{2\eta}\frac{dP}{dr}\right)\cdot\left[z^2 - \frac{b^2}{4}\right] \qquad \ldots\ldots\ldots\ldots\ldots \text{(iv)}$$

$$\therefore \qquad Q = 2\int_{z=0}^{\frac{b}{2}} v(z)\cdot 2\pi r dz = \left(\frac{2\pi r}{\eta}\frac{dP}{dr}\right)\int_{z=0}^{\frac{b}{2}}\left[z^2 - \frac{b^2}{4}\right]dz$$

$$Q = -\frac{\pi r b^3}{6\eta}\frac{dP}{dr} \qquad \ldots\ldots\ldots\ldots\ldots\ldots \text{(v)} \qquad \text{(0.7 point)}$$

3.4) The total rate of heat flow from the area πr^2 of the hot surface to the drop is $\frac{\pi r^2 \mathcal{K}\,\Delta T}{b}$. We assume that this heat goes into vaporizing the drop.

Hence $\quad \rho Q \ell = \frac{\pi r^2 \mathcal{K} \Delta T}{b}$ and using (v) we get

$$\frac{dP}{dr} = -\left(\frac{6\eta \mathcal{K} \Delta T}{\rho_V \ell b^4}\right) \cdot r \qquad \text{.....................} \quad \text{(vi)} \qquad \text{(0.4 point)}$$

This gives $\quad P(r) = -\left(\frac{3\eta \mathcal{K} \Delta T}{\rho_V \ell b^4}\right) \cdot r^2 + B$ (0.4 point)

where B is an arbitrary constant whose value can be found by applying the boundary condition $P(R) = P_a$, the atmospheric pressure.

Hence $\quad B = P_a + \left(\frac{3\eta \mathcal{K} \Delta T}{\rho_V \ell b^4}\right) \cdot R^2$ .... (vii) (0.4 point)

and $\quad P(r) = P_a + \left(\frac{3\eta \mathcal{K} \Delta T}{\rho_V \ell b^4}\right) \cdot \left(R^2 - r^2\right)$ .... (viii) (0.8 point)

3.5) The net force due to pressure is in the upward direction and of magnitude

$$f = \int_{r=0}^{R} \left[P(r) - P_{\mathrm{a}}\right] 2\pi r dr = \frac{3\pi\eta\mathcal{K}\Delta T R^4}{2\rho_{\mathrm{V}}\ell b^4} \quad \text{.........(ix)} \quad \text{(1.0 point)}$$

The weight of the drop is $\frac{2}{3}\pi R^3 \rho_0 g$, where ρ_0 is the density of liquid.

$$\therefore \quad \frac{2}{3}\pi R^3 \rho_0 g = \frac{3\pi\eta\mathcal{K}\Delta T R^4}{2\rho_{\mathrm{V}}\ell b^4}$$

$$b = \left(\frac{9\eta\mathcal{K}R\Delta T}{4\rho_0\rho_{\mathrm{V}}\ell g}\right)^{\frac{1}{4}} \quad \text{..............(x)}$$

Note that $$\frac{3\eta\mathcal{K}\Delta T}{\rho_{\mathrm{V}}\ell b^4} = \frac{4}{3}\frac{\rho_0 g}{R} \quad \text{..............(xi)} \quad \text{(1.0 point)}$$

3.6) Use equations (xi) and (viii) to obtain

$$P(r) = P_{\mathrm{a}} + \left(\frac{4}{3}\frac{\rho_0 g}{R}\right)\cdot\left(R^2 - r^2\right) \quad \text{..............(xii)}$$

$$\frac{d}{dr}P(r) = -\left(\frac{8}{3}\frac{\rho_0 g}{R}\right)\cdot r \quad \text{..............(xiii)} \quad \text{(0.8 point)}$$

Then use (v) to calculate the total mass-rate of vaporization $Q\rho_{\mathrm{V}}$ at $r = R$:

$$Q\rho_{\mathrm{V}} = \left(\frac{2\pi b^3 R}{12\eta}\right)\left(\frac{8}{3}\frac{\rho_0 g}{R}\right)R\rho_{\mathrm{V}} = \left(\frac{4\pi\rho_{\mathrm{V}}\rho_0 g R}{9\eta}\right)b^3$$

$$= \left(\frac{4\pi\rho_{\mathrm{V}}\rho_0 g R}{9\eta}\right)\left(\frac{9\eta\mathcal{K}R\Delta T}{4\rho_0\rho_{\mathrm{V}}\ell g}\right)^{\frac{3}{4}}$$

$$= \left(\frac{4\pi^4\mathcal{K}^3\rho_{\mathrm{V}}\rho_0 g(\Delta T)^3}{9\eta\ell^3}\right)^{\frac{1}{4}}\cdot R^{\frac{7}{4}} = \beta R^{\frac{7}{4}} \quad \text{...... (xiv)} \quad \text{(1.2 points)}$$

3.7) The life-time (τ) of the drop, is to be found from

$$\frac{d}{dt}\left(\frac{2}{3}\pi R^3 \rho_0\right) = -Q\rho_{\mathrm{V}} = -\beta R^{\frac{7}{4}}$$

$$R^{\frac{1}{4}}\frac{d}{dt}R = -\frac{\beta}{2\pi\rho_0}$$

$$\int_R^0 R^{\frac{1}{4}} dR = -\int_0^\tau \frac{\beta}{2\pi\rho_0}dt \qquad \text{(1.0 point)}$$

$$\tau = \frac{8\pi\rho_0}{5\beta}R^{\frac{5}{4}} = \frac{8}{5}\left(\frac{9\eta\rho_0^3\ell^3}{4\mathcal{K}^3\rho_{\mathrm{V}}g\left(\Delta T\right)^3}\right)^{\frac{1}{4}}\cdot R^{\frac{5}{4}} \qquad \text{(1.0 point)}$$

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